On a Multiple Item Selling Model with Vector Offers with Applications to Organizational Hiring and a General Sequential Stochastic Assignment Model

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D-L-R Seq. Stoch. Assignment Problem:

$N = \{1, \ldots, n\}$ is set of people having values $p_1, \ldots, p_n$

Jobs arrive sequentially; job has value x

return is px

General Seq. Stoch. Assignment Problem:

$N = \{1, \ldots, n\}$

Jobs arrive sequentially; $\mathbf{x} = (x_1, \ldots, x_n)$

job can be rejected: C, $0 < \beta \leq 1$

Interpretation; workers for sale; job is a bid.
What if allow bids for multiple workers?

Set of items to sell: $N = \{1, \ldots, n\}$

Buyers bid for specified subsets $S_1, \ldots, S_k$ of items

Bid is a vector $\mathbf{X} = (X_{S_1}, \ldots, X_{S_k})$ with known dist.

At most one of the subsets can be sold to each buyer.

State of the system: $(S, \mathbf{x})$

Optimality equation:

$$V(S,\mathbf{x}) = \max\left(\beta V(S),\ \max_{1\le i\le k: S_i \subset S}[x_{S_i} + \beta V(S - S_i)]\right) - c$$

$$= \max\left(\beta V(S),\ R(S,\mathbf{x})\right) - c$$

$$V(T) = E[V(T,\mathbf{X})]$$

Proposition 1 *$V(S)$ is the unique value v such that*

$$c + (1-\beta)v = E[(R(S,\mathbf{X}) - \beta v)^+] \qquad (1)$$

Numerical Procedure

- Generate iid random offer vectors $\mathbf{X}^j, j = 1, \ldots, m$
- Using $R(\{i\}, \mathbf{x}) = x_{\{i\}}$, determine $V(i), i = 1, \ldots, n$
- Note that this yields $R(S, \mathbf{x})$ for $|S| = 2$
- Determine $V(S)$ for all two point sets S

How to determine $V(S)$ when know $V(T)$ for $T \subset S$

binary search

$$E[(R(S, \mathbf{X}) - \beta v^*)^+] \approx \Sigma_{j=1}^m \frac{1}{m} \left(R(S, \mathbf{X}^j) - \beta v^*\right)^+.$$

Proposition 2

$$V(S) \geq \max_{1 \leq i \leq k : S_i \subset S} [E[X_{S_i}] + \beta V(S - S_i)] - c$$

1 A Special Case Model where Buyers bid for all Subsets

Offer vector $Y_1, \ldots, Y_n$
Buyer willing to buy any set T for the price $\Sigma_{i \in T} Y_i$.

$$\begin{aligned} V(S, \mathbf{y}) &= \max_{\emptyset \subset S' \subset S} [\sum_{j \in S'} y_j + \beta V(S - S')] - c \\ &= \max(\beta V(S),\ R(S, \mathbf{y})) - c \end{aligned}$$

B-F gave OE, but not its solution.

Let $\alpha_i(c) \equiv \beta V(\{i\})$.

Proposition 3 *Optimal policy never sells i for the offered value $y_i < \alpha_i(c)$.*

Should you always sell i in state $(S, \mathbf{y})$ if $y_i > \alpha_i(c/|S|)$?

Example: $n = 2,\ \beta = c = 1$. Suppose Y_1, Y_2 ind.

$$P(Y_1 = 1) = .99, \quad P(Y_1 = 10) = .01$$

$$P(Y_2 = 1) = 1 - 10^{-10}, \quad P(Y_2 = 10^{20}) = 10^{-10}$$

Proposition 4 *It is optimal in state* $(S, \mathbf{y})$ *to sell all items in* S *when* $y_i \geq \alpha_i(c/|S|)$ *for all* $i \in S$.

Proposition 5 *For* $|S| \geq 2$,

$$\max_{i \in S}\{\alpha_i(c)/\beta + V(S-i)\} \leq V(S) \leq \sum_{i \in S} \alpha_i(c/|S|)/\beta$$

If it is optimal to sell item $1 \in S$ when the offer vector is $x_1, \ldots, x_n$ it is necessarily optimal to sell item 1 if the offer vector were $y_1, \ldots, y_n$ whenever $y_1 > x_1$?

$P(Y_1 = 1) = .98, P(Y_1 = 2) = .01, P(Y_1 = 10) = .01$

Optimal to sell item 1 (and item 2) if the offer vector were $(1, 10^{20})$ but optimal to sell neither if the offer vector were $(2, 1)$.

However, if optimal to sell 1 when the offer vector is $x_1, \ldots, x_n$ then optimal to sell 1 if the offer vector were $y_1, \ldots, y_n$ provided that $y_i \geq x_i$ for all $i \in S$.

That is, optimal set to sell is increasing function of offer vector

Lemma 1 *(Bruss-Ferguson)*

$$V(S \cup T) + V(S \cap T) \geq V(S) + V(T)$$

New Proof:

Seller 1 has $|S| + |T|$ items
arranges in 2 collections: S and T

Seller 2 has same $|S| + |T|$ items
arranges in 2 collections: $S \cup T$ and ST

offer vector $y_1, \ldots, y_n$, buyer will buy any number
sellers cost per period: c per unsold collection
seller 1 uses opt. policy: yields $V(S) + V(T)$
seller 2 matches 1 always choosing from ST collection

Proposition 6 *In state* $(S, \mathbf{y})$, *the optimal set*
(a) sold is an increasing function of $\mathbf{y}$.
(b) not sold is an increasing function of S.

A Heuristic Policy when n is Large

Problem with $2n$ items. Randomly partition into two sets N_1 and N_2 of n items each. Use optimal policies in subproblems with per period cost $c/2$. Recombine when possible.

Example 2. $n = 2,\ X_1, X_2$ are ind $(0, 1),\ \beta = 1$.

Table 1: For $n = 2$

c	expected return from heuristic policy	optimal expected retu
.1	1.271	1.273
.2	.996	1.000
.3	.799	.804
.4	.644	.651
.5	.519	.524
.6	.409	.412
.7	.304	.305
.8	.201	.201
.9	.100	.100

Offer Vectors are iid

Prop. $V(r)$ is convex.

Table 2: $V(n)$ when $X_1, \ldots, X_n$ are iid uniform $(0, 1)$ and $c = .1$

n	$V(n)$
1	0.553
2	1.273
3	2.035
4	2.826
5	3.637
6	4.463
7	5.302
8	6.150
9	7.007
10	7.871